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On Local Implicit Approximation and Its Applications

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1 Introduction

A recurring operation in solid modeling is the evaluation of surface intersections [14]. If both surfaces are given parametrically, two major approaches are subdivision, and substitution.

In the subdivision method [5,7,8,9,13], both surfaces are recursively subdivided in the vicinity of their intersection. The subdivision results in an adaptive piecewise linear approximation of both surfaces and their intersection. Among the advantages of the method we mention its robustness and its potential for locating all intersection branches. A major drawback of the subdivision method is the large volume of data it creates, which slows it down in areas of high surface curvature.

In the substitution method [3,10,15], one of the surfaces, S_1 is converted to implicit form F, and the parametric form of S_2 is substituted into F resulting in an implicit algebraic curve f in the parameter space of S_2. This curve f is in birational correspondence with the intersection of S_1 and S_2 in xyz-space, thus serves as an accurate representation of the intersection. Major difficulties of the substitution method limit its utility in practice. There are two general methods for implicitizing a parametric surface. The first method is based on Elimination Theory [16] and does resultant computations. It is expensive and generates extraneous factors whose detection is a delicate problem. The second method for implicitization is based on Gröbner Basis techniques [2]. It is also fairly expensive and requires, moreover, rational coefficients in the description of S_1. Another difficulty with the substitution method, less prominently pointed out but well-known [12], is that the substitution itself can be numerically unstable, and is a nontrivial algorithmic task when desiring efficiency and accuracy. Some authors have suggested the use of rational arithmetic for this reason [4], thus further adding to the computational load of the approach.

In view of these difficulties, we seek an alternative that, on the one hand avoids generating a space intensive linear approximations, and, on the other hand, side-steps expensive computations associated with implicitization and substitution into high-degree implicit forms.

In this paper, we explore the utility of locally approximating parametric curves and surfaces with low-degree implicit forms. These approximations could be constructed as part of a subdivision process, although some aspects of this remain to be investigated. Having constructed a local implicit approximation, a substitution of the parametric form of the other surface could be done, or the intersection could be evaluated from implicit forms [1,6].

Local explicit approximations to parametric curves and surfaces have also been proposed in [11]. There, an approximation of the form

$$z = f(x,y) = \sum a_{ij} x^i y^j \quad \text{or} \quad y = f(x) = \sum a_i x^i$$

is constructed, for surfaces and curves. Recurrence formulas were also derived for the coefficients of f. In our experience, a locally explicit function is less favorable than a local implicit approximation. In fact, while a quadratic explicit approximation to a curve achieves second order contact at the point at which it is constructed, a quadratic implicit approximation achieves fifth order contact. For curves, the order of contact grows linearly with the degree of the explicit approximation, whereas the implicit approximation has a quadratic growth in the order of contact. Thus, much lower degree approximations suffice. Moreover, local explicit approximation in general can only approximate curves or surfaces *locally* no matter how high the degree is while an local implicit approximation is capable of approximating curves or surfaces not only locally but also globally in the sense that the radius of convergence increases when the degree of approximation increases, and the exact implicitization can be finally derived when the degree of approximation is equal to the degree of the given parametric curve or surface.

2 Preliminaries

We denote a polynomial of degree n in $\mathrm{u} = [x_1, x_2, \ldots, x_k]$ by $f^n(\mathrm{u})$. Partial derivatives of f are written by subscripting, e.g. $f_x = \partial f/\partial x$. The *gradient* of f at the point $\mathrm{u} = [x_1, x_2, \ldots, x_k]$ is the vector $\nabla f = (f_{x_1}, f_{x_2}, \ldots, f_{x_k})$, where the partials are evaluated at u.

A rational plane curve $r(t)$ can be given as the pair $(x(t), y(t))$, where $x(t)$ and $y(t)$ are rational functions of t. The curve points are all points $(x(t), y(t))$ on the plane. The curve is *properly parameterized* if for all but finitely many curve points p we have $p = (x(t), y(t))$ with a unique value of t; otherwise, the parameterization is *improper* and there is a rational nonlinear function $s(t)$ such that $x(t) = x^*(s(t))$ and $y(t) = y^*(s(t))$. We assume here that all curves specified parametrically are properly parameterized and note that a properly parameterized curve is always irreducible. For methods to detect improper parameterization, see Sederberg [17].

The degree of a rational parametric curve is the highest degree of the enumerator or the denominator polynomial, assuming both $x(t)$ and $y(t)$ have been written with a common denominator. The implicit equation $f(x, y)$ of the rational curve $r(t)$ is a smallest degree polynomial in x and y satisfying $f(x(t), y(t)) \equiv 0$. It is unique up to a multiplicative constant. If $r(t)$ has degree m, then so does $f(x, y)$.

As with parametric curves, a parametric surface

$$P(s,t) = (x(s,t), y(s,t), z(s,t))$$

can be *improperly parameterized* if there are nonlinear polynomials $u(s,t)$

and $v(s,t)$ such that

$$P(s,t) = (x^*(u(s,t),v(s,t)), y^*(u(s,t),v(s,t)), z^*(u(s,t),v(s,t)))$$

In that case, there is a many-to-one correspondence between the parameter values and the surface points. $P(s,t)$ is *properly parameterized* if this correspondence is one-to-one except, possibly, on a one dimensional set of points. We also assume that all parametric surfaces are properly parameterized. For a parametric surface described by rational functions of degree m, there always exists an irreducible implicit equation $f(x,y,z) = 0$ satisfying $f(x(s,t), y(s,t), z(s,t)) \equiv 0$, and f is unique within a constant factor. Moreover, f has degree at most m^2.

A point $p = (x,y)$ is *regular* on a plane curve $f(x,y) = 0$ if the gradient of f at p is not null; otherwise the point is *singular*. Likewise, $p = (x,y,z)$ is regular on $f(x,y,z) = 0$ if the gradient of f is not null at p, otherwise p is singular.

3 Local Implicit Approximation of Parametric Plane Curves

Let

$$r(t) = (x(t), y(t)) = (\frac{p(t)}{w(t)}, \frac{q(t)}{w(t)})$$

be a rational parametric plane curve of degree m containing the origin, where

$$p(t) = \sum_{i=1}^{m} a_i t^i, \qquad q(t) = \sum_{i=1}^{m} b_i t^i, \qquad w(t) = \sum_{i=0}^{m} c_i t^i$$

with a_m and b_m not both zero and $c_0 \neq 0$. There always exists an irreducible polynomial $f^m(x,y) = 0$ of degree m representing $r(t)$, provided $r(t)$ is properly parameterized. The polynomial $f^m(x,y)$ can be found by eliminating t from the equations

$$x\,w(t) - p(t) = 0, \qquad y\,w(t) - q(t) = 0$$

We look for an implicit form of degree n that approximates the curve $r(t)$ locally at the origin and has degree $n < m$.

Montaudouin et al. [11] proposed a local explicit approximation to $r(t)$ at the origin, of the form

$$y = h(x)$$

Assuming that $r(t)$ is integral, i.e. $w(t) = 1$, and that $dx/dt \neq 0$ at the origin, a computation is described for deriving $h(x)$ of degree n. The order of contact is q, when

$$y(t) - h(x(t)) = \sum_{i>q}^{nm} d_i t^i$$

and is, in general, at most the degree of h.

We propose a local implicit approximation to $r(t)$ of the form

$$g(x, y) = 0$$

with g having degree $n < m$. We will see that this method works both for integral and rational parametric curves, that it achieves higher order of contact at the origin, and that it has better global behavior.

3.1 Derivation of the Local Implicit Approximation

Let $g^n(x, y) = \sum_{i+j=1}^{n} e_{ij} x^i y^j = 0$ be a degree n implicit curve containing the origin. We wish to determine the coefficients e_{ij}. Since $g^n(x, y) = 0$ and $\gamma g^n(x, y) = 0$, $\gamma \neq 0$, are the same curve , we can choose one of the e_{ij} as 1 and have to determine $(n^2 + 3n - 2)/2$ additional coefficient values. Thus $g^n(x, y) = 0$ has degree of freedom $\varphi(n) = (n^2 + 3n - 2)/2$. Let $G^n(x, y, z)$ be the homogeneous form of $g^n(x, y)$. We set

$$g^n(\frac{p(t)}{w(t)}, \frac{q(t)}{w(t)}) = \frac{G^n(p(t), q(t), w(t))}{(w(t))^n} = \frac{\sum_{i=1}^{nm} \alpha_i^n t^i}{\sum_{i=0}^{nm} \beta_i t^i}$$

The α_i^n are linear combinations of the e_{ij}. We require that at least the first $\varphi(n)$ powers of t vanish in $G^n(p(t), q(t), w(t))$, from which, in principle, we can determine the values for e_{ij}. To do so, we explain first how to find the linear system entailed by the α_i^n, and then show how to solve for the e_{ij}.

3.1.1 Computing α_i^n

Let α_i^{n-1} denote the coefficient of t^i in $G^{n-1}(p(t), q(t), w(t))$. Then

$$\alpha_i^n = \textit{coefficient of } t^i \textit{ in } (w(t) \sum_{j=1}^{m(n-1)} \alpha_j^{n-1} t^j + \sum_{k+l=n} e_{kl}(p(t))^k (q(t))^l)$$

Setting

$$(p(t))^k = (\sum_{i=1}^{m} a_i t^i)^k = \sum_{i=k}^{km} (a(k))_i t^i$$

and similarly,

$$(q(t))^l = (\sum_{i=1}^{m} b_i t^i)^l = \sum_{i=l}^{lm} (b(l))_i t^i$$

we obtain that

$$\alpha_i^n = \sum_{j=1}^{i} \alpha_j^{n-1} c_{i-j} + \sum_{k+l=n} \sum_{p+q=i} e_{kl}(a(k))_p (b(l))_q$$

in particular, $\alpha_i^1 = c_{10} a_i + c_{01} b_i$.

For an integral parametric curve $r(t)$, the α_i^n specialize to

$$\alpha_i^n = \begin{cases} \alpha_i^{n-1} & 1 \le i \le n-1 \\ \alpha_i^{n-1} + \sum_{k+l=n} \sum_{p+q=i} e_{kl}(a(k))_p (b(l))_q & n \le i \le (n-1)m \\ \sum_{k+l=n} \sum_{p+q=i} e_{kl}(a(k))_p (b(l))_q & (n-1)m < i \le nm \end{cases}$$

Note that $(a(k+1))_i$ can be computed, assuming the $(a(k))_k, (a(k))_{k+1}, \ldots, (a(k))_{i-1}$ are known, see [11]. $(b(l))_i$ are computed analogously.

3.1.2 Computing the Local Implicit Approximation of Curves

We now show that the coefficient matrix of the linear system entailed by the α_i^n has rank at least $\varphi(n)$, and that there are nontrivial solutions to it. To this end, we consider initial segments of the exact implicit form $f^m(x, y)$ of $r(t)$. Note, however, that an algorithmic determination of these initial segments achieves a lower order contact than that obtained by other implicit forms of equal degree.

Let $\mathbf{r}_i^n$ be the coefficient vector of α_i^n, $i = 1, 2, \ldots, nm$, such that $\mathbf{r}_i^n \cdot \mathbf{e}_n = \alpha_i^n$, where $\mathbf{e}_n = (e_{10}, e_{01}, e_{20}, e_{11}, e_{02}, \ldots, e_{n0}, e_{(n-1)\,1}, \ldots, e_{1\,(n-1)}, e_{0n})^T$. Let

A_{mn} be defined as $(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{nm})^T$. Hence $A_{mn}\mathbf{e}_n = (\alpha_1^n, \alpha_2^n, \ldots, \alpha_{nm}^n)^T$ and A_{mn} is of dimension nm by $\varphi(n)+1$. Furthermore, the maximum rank of A_{mn} is $\varphi(n)+1$ since $m \geq n$ and $nm \geq \varphi(n)+1$. Example 3.1 shows matrix A_{32} symbolically.

Example 3.1: For $m = 3$ and $n = 2$, A_{32} is

$$\begin{pmatrix} a_1c_0 & b_1c_0 & 0 & 0 & 0 \\ a_1c_1 + a_2c_0 & b_1c_1 + b_2c_0 & a_1^2 & a_1b_1 & b_1^2 \\ a_1c_2 + a_2c_1 + a_3c_0 & b_1c_2 + b_2c_1 + b_3c_0 & 2a_1a_2 & a_1b_2 + a_2b_1 & 2b_1b_2 \\ a_1c_3 + a_2c_2 + a_3c_1 & b_1c_3 + b_2c_2 + b_3c_1 & 2a_1a_3 + a_2^2 & a_1b_3 + a_2b_2 + a_3b_1 & 2b_1b_3 + b_2^2 \\ a_2c_3 + a_3c_2 & b_2c_3 + b_3c_2 & 2a_2a_3 & a_2b_3 + a_3b_2 & 2b_2b_3 \\ a_3c_3 & b_3c_3 & a_3^2 & a_3b_3 & b_3^2 \end{pmatrix}$$

□

When computing the local implicit approximation $g^n(x,y)$ of $r(t)$, if the rank of A_{mn} is at least $\varphi(n)$ then we can select one coefficient of $g^n(x,y)$ to be 1 and determine the others by requiring $\alpha_{i_1}^n = \alpha_{i_2}^n = \cdots = \alpha_{i_{\varphi(n)}}^n = 0$, where the corresponding $(\mathbf{r}_{i_1}, \mathbf{r}_{i_2}, \ldots, \mathbf{r}_{i_{\varphi(n)}})$ are the first $\varphi(n)$ linearly independent row vectors from A_{mn}.

Let $f^m(x,y) = 0$ be the exact irreducible implicit form of $r(t)$ with $F^m(x,y,z)$ as its corresponding homogeneous form, and let f^n, $n < m$, be the degree n initial segment of f^m with its corresponding homogeneous form $F^n(x,y,z)$, that is, $f^m(x,y) = f^n(x,y) + \textit{terms with degree} > n$. Also, let $\sum_{i=1}^{nm} \bar{b}_i t^i \equiv F^n(p(t), q(t), w(t))$ and $\bar{\mathbf{b}}_{mn} \equiv (\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_{mn})^T$.

Lemma 3.1 *With notations defined above, we have*
1. $f^n(x,y)$ *is the zero polynomial if and only if* $\bar{\mathbf{b}}_{mn} = 0$.
2. if $f^n(x,y)$ *is a nonzero polynomial then* $\bar{b}_1 = \bar{b}_2 = \cdots = \bar{b}_n = 0$ *and* $\bar{\mathbf{b}}_{mn} \neq 0$.

Proof: part 1: "$\Rightarrow$" trivial.
"$\Leftarrow$" Suppose $F^n(x,y)$ is a nonzero polynomial, since $\bar{\mathbf{b}}_{mn} = 0$,

$$F^n(p(t), q(t), w(t)) = 0 = (w(t))^n f^n\left(\frac{p(t)}{w(t)}, \frac{q(t)}{w(t)}\right)$$

for all t and then $f^n(p(t)/w(t), q(t)/w(t)) = 0$, for all t with possibly finitely many exceptions, where $w(t) = 0$. Thus $f^n(x,y)$ with $n < m$ also represents $r(t)$ which contradicts the irreducibility of $f^m(x,y)$.

part 2: Since $f^m(x,y) = 0$ is the implicit form of $r(t)$,

$$f^m(\frac{p(t)}{w(t)}, \frac{q(t)}{w(t)}) = 0$$

for all t except finitely many t where $w(t) = 0$. Thus

$$F^m(p(t), q(t), w(t)) = (w(t))^m f^m(\frac{p(t)}{w(t)}, \frac{q(t)}{w(t)}) = 0$$

for all t. From

$$F^n(p(t), q(t), w(t)) + \sum_{i+j=n+1}^{m} e_{ij}(p(t))^i(q(t))^j(w(t))^{m-i-j} = 0$$

for every t, we have

$$\sum_{i=1}^{nm} \bar{b}_i t^i = - \sum_{i+j=n+1}^{m} e_{ij}(p(t))^i(q(t))^j(w(t))^{m-i-j} = \sum_{i=n+1}^{nm} \bar{b}'_i t^i$$

for every t. By comparison, we have $\bar{b}_1 = \bar{b}_2 = \cdots = \bar{b}_n = 0$. The rest of part 2 follows from part 1. □

As an initial segment of $f^m(x,y)$, $f^n(x,y)$ could be either a zero polynomial or a nonzero polynomial with zero or nonzero $\bar{\mathrm{b}}_{mn}$ vectors respectively. If $\bar{\mathrm{b}}_{mn}$ is known beforehand, the coefficient vector e_n of $f^n(x,y)$ is uniquely determined by $\mathrm{A}_{mn}\mathrm{e}_n = \bar{\mathrm{b}}_{mn}$ which is an overdetermined linear system. Note that, for a fixed n, the elements of the matrix A_{mn} depend only on the coefficients of $p(t), q(t)$, and $w(t)$. The following results characterize the rank of A_{mn}.

Lemma 3.2 *If $r(t)$ is a properly parameterized rational parametric plane curve with irreducible implicit form $f^m(x,y) = 0$ then for $n < m$, we have*

$$rank(\mathrm{A}_{mn}) = \varphi(n) + 1$$

Proof: Suppose, knowing $\bar{\mathrm{b}}_{mn}$, we want to determine the coefficients of $f^n(x,y)$ by solving the overdetermined linear system $\mathrm{A}_{mn}\mathrm{e}_n = \bar{\mathrm{b}}_{mn}$, where e_n is the coefficient vector of the general degree n polynomial.

If $\bar{\mathrm{b}}_{mn} = 0$ then $\mathrm{A}_{mn}\mathrm{e}_n = 0$ is a homogeneous system. If $rank(\mathrm{A}_{mn}) < \varphi(n) + 1$, there will be infinitely many nontrivial solutions as well as the trivial solution for this linear system. This cannot be true by Lemma 3.1. Thus $rank(\mathrm{A}_{mn}) = \varphi(n) + 1$ if $\bar{\mathrm{b}}_{mn} = 0$.

If $\bar{\mathrm{b}}_{mn} \neq 0$ then $\mathrm{A}_{mn}\mathrm{e}_n = \bar{\mathrm{b}}_{mn}$ is a *consistent* non-homogeneous system since there is always a solution, that is the coefficient of $f^n(x,y)$. Suppose $rank(\mathrm{A}_{mn}) < \varphi(n) + 1$, this system will have infinitely many solutions. Let e_n^* be one of the infinitely many solutions and $\mathrm{e}_n^* \neq \mathrm{e}_n$, where e_n is the coefficient vector of $f^n(x,y)$. Let also $h^n(x,y)$ be the corresponding polynomial of e_n^* and

$$h^m(x,y) \equiv h^n(x,y) + \textit{terms of } f^m(x,y) \textit{ with degree} > n$$

Let $H^m(x,y,z)$ and $H^n(x,y,z)$ be the homogeneous polynomials of $h^m(x,y)$ and $h^n(x,y)$ respectively. Since $\mathrm{A}_{mn}\mathrm{e}_n = \mathrm{A}_{mn}\mathrm{e}_n^* = \bar{\mathrm{b}}_{mn}$,

$$H^n(p(t),q(t),w(t)) = F^n(p(t),q(t),w(t))$$

for every t, and thus

$$H^m(p(t),q(t),w(t)) = F^m(p(t),q(t),w(t)) = 0$$

for all t. We then have

$$f^m\left(\frac{p(t)}{w(t)},\frac{q(t)}{w(t)}\right) = h^m\left(\frac{p(t)}{w(t)},\frac{q(t)}{w(t)}\right) = 0$$

for all but finitely many t. Hence $f^m(x,y)$ and $h^m(x,y)$ represent the same algebraic curve. Since $f^n(x,y) \neq h^n(x,y)$, $f^m(x,y)$ and $h^m(x,y)$ differ by more than a constant factor which contradicts that the equation of an irreducible curve is unique to within a constant factor. Therefore $rank(\mathrm{A}_{mn})$ must be $\varphi(n) + 1$ if $\bar{\mathrm{b}}_{mn} \neq 0$. □

Lemma 3.3 *If $r(t)$ is a properly parameterized rational parametric plane curve with irreducible implicit form $f^m(x,y) = 0$ then for $n = m$, we have*

$$rank(\mathrm{A}_{mm}) = \varphi(m)$$

Proof: If $n = m$, $f^m(p(t)/w(t), q(t)/w(t)) = 0$ for all t except finitely many t where $w(t) = 0$, and then $F^m(p(t), q(t), w(t)) = 0$ for all t, we thus have $\mathrm{A}_{mm}\mathrm{e}_m = 0$ which is an overdetermined linear homogeneous system. Since we will have only trivial solution if $rank(\mathrm{A}_{mm}) = \varphi(m) + 1$, $rank(\mathrm{A}_{mm})$ must be less than or equal to $\varphi(m)$.

Suppose $r \equiv rank(\mathrm{A}_{mm}) < \varphi(m)$, then the solution space of the overdetermined homogeneous system has as basis $p \equiv \varphi(m) + 1 - r$ linearly independent vectors and every solution of this system is the linear combination of these p solutions. Now suppose that $r < \varphi(m)$, then the system has a solution space spanned by $p > 2$ linearly independent vectors, say $\mathrm{e}_m^1, \mathrm{e}_m^2, \cdots, \mathrm{e}_m^p$. Let $f_i^m(x, y)$ be the corresponding polynomial with coefficient vector e_m^i and $F_i^m(x, y, z)$ be the homogeneous form of $f_i^m(x, y)$, $i = 1, 2, \ldots, p$. Since

$$F_i^m(p(t), q(t), w(t)) = 0 = (w(t))^m f_i^m(\frac{p(t)}{w(t)}, \frac{q(t)}{w(t)})$$

for all t, $1 \leq i \leq p$, we have $f_i^m(p(t)/w(t), q(t)/w(t)) = 0$ for all t with finitely many exceptions where $w(t) = 0$, $1 \leq i \leq p$. Thus the irreducible curve $f^m(x, y) = 0$ can be represented by $f_i^m(x, y), i = 1, 2, \ldots, p$, which are not different within only a constant factor because $\mathrm{e}_m^1, \mathrm{e}_m^2, \cdots, \mathrm{e}_m^p$ are linearly independent. By the above arguments, we can conclude that $rank(\mathrm{A}_{mm}) = \varphi(m)$. □

By assigning one variable to be 1, the nontrivial solution of $\mathrm{A}_{mm}\mathrm{e}_m = 0$ is guaranteed by Lemma 3.3, and it is the coefficient vector of the exact implicitization of $r(t)$. The previous Lemmas can be summarized as follows:

Theorem 3.4 *If $r(t)$ is a properly parameterized rational parametric plane curve of degree m then*

$$rank(\mathrm{A}_{mn}) = \begin{cases} \varphi(n) + 1 & \text{if } n < m \\ \varphi(n) & \text{if } n = m \end{cases}$$

As the result of Theorem 3.4, we compute the degree n local implicit approximation for rational parametric plane curves defined by rational functions of degree m as follows: We suppose the origin is nonsingular. Let $\alpha = 0$ be either $e_{10} - 1 = 0$ or $e_{01} - 1 = 0$, depending on $\dot{y}(0) \neq 0$ or $\dot{x}(0) \neq 0$. Let r be the coefficient vector of $\alpha = 0$ with respect to e_n, and consider the linear system $\mathrm{B}_{mn}\mathrm{e}_n = \mathrm{b}_{mn}$, where $\mathrm{B}_{mn} = (\mathrm{r}, \mathrm{r}_1, \cdots, \mathrm{r}_{mn})^T$, and $\mathrm{b}_{mn} = (1, 0, \cdots, 0)^T$. Let $\mathrm{B}_{mn}^i\mathrm{e}_n = \mathrm{b}_{mn}^i$ be the subsystem which consists of the first $i + 1$ equations of $\mathrm{B}_{mn}\mathrm{e}_n = \mathrm{b}_{mn}$. The coefficients of $g^n(x, y)$

can then be solved by

$$\mathrm{B}_{mn}^{k}\mathrm{e}_n = \mathrm{b}_{mn}^{k}$$

where $k \geq \varphi(n)$ is the smallest integer such that B_{mn}^{k} is of rank $\varphi(n)+1$ and $\mathrm{B}_{mn}^{k}\mathrm{e}_n = \mathrm{b}_{mn}^{k}$ is a consistent system. This computation may result in a l-th order of contact at the origin, where $l \geq k \geq \varphi(n)$, since $\alpha_{k+1}^{n}, \cdots, \alpha_{l}^{n}$ may all vanish on the solution so derived, i.e. the solution actually satisfies $\mathrm{B}_{mn}^{l}\mathrm{e}_n = \mathrm{b}_{mn}^{l}$.

Next, we show that the $g^n(x,y)$ of $r(t)$ at nonsingular point $(0,0)$ is irreducible if it is computed by the consistent system $\mathrm{B}_{mn}^{k}\mathrm{e}_n = \mathrm{b}_{mn}^{k}$. Note that the local implicit approximations so derived have the same linear terms if $\alpha = 0$ in the systems $\mathrm{B}_{mn}^{k}\mathrm{e}_n = \mathrm{b}_{mn}^{k}$ are the same.

Lemma 3.5 *Let $l(n)$ be the order of contact made by the degree n local implicit approximation $g^n(x,y)$. If $g^n(x,y)$ and $g^{n-1}(x,y)$ of $r(t)$ at nonsingular point $(0,0)$ are computed by augmenting the same $\alpha = 0$ to the system, where $\alpha = 0$ is either $c_{10} - 1 = 0$ or $c_{01} - 1 = 0$, then we have $l(n-1) < l(n)$.*

Proof: Let $\bar{g}^n(x,y) = \frac{1}{2}(g^n(x,y) + g^{n-1}(x,y))$. Suppose $l(n-1) \geq l(n)$, the $\bar{g}^n(x,y)$ so defined has the following four properties: (1) $\bar{g}^n$ is of degree n. (2) $\bar{g}^n \neq g^n$. (3) $\bar{g}^n$ has the same linear terms as g^n since g^n and g^{n-1} have the identical linear terms. (4) $\bar{g}^n$ has the order of contact larger than or equal to $l(n)$ since $l(n-1) \geq l(n)$ is assumed. From properties 1, 3, and 4, the coefficients of $\bar{g}^n$ satisfy the linear system that is used to compute the coefficients of g^n; but property 2 contradicts the uniqueness of solution of a nonsingular linear system. Thus $l(n-1) < l(n)$. □

Lemma 3.6 *With $l(n)$ defined in Lemma 3.5, $l(n') < l(n)$, for $n' < n$.*

Proof: Since $r(0)$ is not singular, augmenting with the same $\alpha = 0$ cannot give rise to inconsistencies. Hence, by induction and Lemma 3.5, $l(n)$ is strictly monotone. □

Lemma 3.7 *At the nonsingular point $(0,0)$, the degree n local implicit approximation $g^n(x,y)$ of the degree $m > n$ properly parameterized parametric curve $r(t) = (x(t), y(t))$ is irreducible.*

Proof: Suppose $g^n(x,y)$ is reducible, and $g^n = g^k g^l$, where $n = k + l$ and $k, l > 0$. Since g^n contains linear terms, one of the g^k and g^l must have a constant term. Let $g^k(x,y) = \sum_{i+j=1}^{k} p_{ij}x^i y^j$, where p_{10} and p_{01} are not both zero, and $g^l(x,y) = \sum_{i+j=0}^{l} q_{ij}x^i y^j$, where $q_{00} \neq 0$. Let also that $g^n(x(t),y(t)) = \sum_{i=1}^{mn} \alpha_i t^n$, $g^k(x(t),y(t)) = \sum_{i=1}^{mk} \beta_i t^n$, and $g^l(x(t),y(t)) = \sum_{i=0}^{ml} \gamma_i t^n$, where $\gamma_0 = q_{00}$. We thus have

$$\sum_{i=1}^{mn} \alpha_i t^i = (\sum_{i=1}^{mk} \beta_i t^i)(\sum_{i=0}^{ml} \gamma_i t^i)$$

The coefficients of $g^n(x,y)$ are computed by solving the nonsingular system $\mathrm{B}^k_{mn}\mathrm{e}_n = \mathrm{b}^k_{mn}$, for some $k \geq \varphi(n)$. Moreover, $l(n)$, the order of contact of g^n, is greater than or equal to k. Thus the coefficients of g^n satisfy $\mathrm{B}^{l(n)}_{mn}\mathrm{e}_n = \mathrm{b}^{l(n)}_{mn}$. The $\mathrm{B}^{l(n)}_{mn}\mathrm{e}_n = \mathrm{b}^{l(n)}_{mn}$ can be represented in terms of β_i and γ_i as follows, assuming the first equation in the system is $e_{10} = q_{00}p_{10} = 1$,

$$\begin{cases} q_{00}p_{10} = 1 \\ q_{00}\beta_1 = 0 \\ q_{00}\beta_2 + \beta_1\gamma_1 = 0 \\ q_{00}\beta_3 + \beta_2\gamma_1 + \beta_1\gamma_2 = 0 \\ \vdots \\ q_{00}\beta_{l(n)} + \beta_{l(n)-1}\gamma_1 + \cdots + \beta_1\gamma_{l(n)-1} = 0 \end{cases}$$

which implies $q_{00}p_{10} = 1$ and $\beta_1 = \beta_2 = \cdots = \beta_{l(n)} = 0$. Thus g^k has either order of contact larger than or equal to $l(n)$ if $l(n) < km$, or $g^k(x(t),y(t)) = 0$ for all t if $l(n) \geq km$. The first result contradicts Lemma 3.6, and the second contradicts the irreducibility of the exact implicitization of $r(t)$. Thus g^n is irreducible. □

There may be cases that $\mathrm{B}^k_{mn}\mathrm{e}_n = \mathrm{b}^k_{mn}$ is an inconsistent system in which the augmented matrix $[\mathrm{B}^k_{mn}, \mathrm{b}^k_{mn}]$ is of rank $\varphi(n) + 2$, for instance on computing g^n of $r(t) = (t, t^9)$. If this happens, some equations should be removed from the system to ensure the consistency, and the order of contact is less than k.

3.2 Error Analysis

Let $T(\epsilon, n) > 0$ represent a bound on t for a given ϵ such that for $|t| < T$ the orthogonal distance $d(t,n)$ between point $(x(t),y(t))$ and the degree

n approximation $g^n(x,y) = 0$ of $r(t)$ is less than ϵ. The distance $d(t_p, n)$ from a point $P = (x_p, y_p) = (x(t_p), y(t_p))$ on the curve $r(t)$ to the degree n approximation $g^n(x,y) = 0$ is the solution of a difficult nonlinear system. A reasonable estimate of $d(t_p, n)$ would be the distance to the $g^n(x,y) = 0$ in a direction orthogonal to the level curve $g^n(x,y) = c$, where $c = g^n(x_p, y_p)$, denoted by $d'(t_p, n)$. Note that $d'(t,n) \geq d(t,n)$ since $d(t,n)$ is the shortest distance from the point to the curve. Let $P' = (x'_p, y'_p)$ be the point on $g^n(x,y) = 0$ on which $g^n(x,y) = 0$ intersects the line orthogonal to level curve $g^n(x,y) = c$ at P, see Figure 3.1. The Taylor series on $P' = (x'_p, y'_p)$ with respect to P is

$$g^n(x'_p, y'_p) = g^n(x_p, y_p) + d'(t_p, n) \cdot \nabla g^n(x_p, y_p) + \textit{higher order terms}$$

Taking the linear term, since $g^n(x'_p, y'_p) = 0$, $d'(t_p, n)$ can be approximated by $d''(t_p, n)$ where

$$d''(t_p, n) = \frac{g^n(x_p, y_p)}{||\nabla g^n(x_p, y_p)||} = \frac{g^n(x(t_p), y(t_p))}{[(g^n_x(x(t_p), y(t_p)))^2 + (g^n_y(x(t_p), y(t_p)))^2]^{1/2}}$$

Note that $d''(t,n)$ may be less than, greater than, or equal to $d(t,n)$ although $d'(t,n)$ is always greater or equal to $d(t,n)$.

We have found no method for computing $T(\epsilon, n)$ analytically. However, in practice we only need a method of obtaining a reasonably good estimate of $T(\epsilon, n)$. Thus it is desirable to determine $T'(\epsilon, n)$, for given ϵ and n, such that $d''(t,n) \leq \epsilon$ for $|t| \leq T'(\epsilon, n)$.

Since $2ab \leq a^2 + b^2$ for any a and b, we have $\sqrt{|a|\,|b|} \leq \frac{|a|+|b|}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$, so that

$$d''(t,n) \leq \hat{d}(t,n) \equiv \frac{\sqrt{2} g^n(x(t), y(t))}{|g^n_x(x(t), y(t))| + |g^n_y(x(t), y(t))|}$$

When tracing $r(t)$, we can detect the first value of t such that $\hat{d}(t,n) \leq \epsilon$ and $\hat{d}(t + \Delta t, n) > \epsilon$, where Δt is the step distance for t.

3.3 Experiments

Below are several examples of both local explicit and implicit approximations.

Example 3.2 Four curve examples are shown below.

$r_1(t) = (t^6 + t^5 - 2t^3 + 3t^2 + 12t, t^6 - t^5 + t^4 - 4t^3 - 2t^2 + 24t)$
$r_2(t) = (3t^6 - 4t^5 - 8t^3 + 6t^2 + 3t, -3t^6 + 4t^5 + 5t^4 - 6t^3 - 8t^2 + 3t)$
$r_3(t) = (3t^6 + t^5 - 2t^4 + 38t^3 - 5t^2 - 14t, t^6 - 12t^5 - 2t^4 + 2t^3 - 7t^2 + 13t)$
$r_4(t) = ((t^6 + 3t^5 - 6t^4 + 4t^3 - 36t^2 + 36t)/w(t), (3t^6 + t^5 - 2t^4 + 39t^3 - 69t^2 + 33t)/w(t))$, where $w(t) = 7t^6 + 10t^5 + 9t^4 + 6t^2 + 3t + 7$.

The curves of $r_1(t)$, $r_2(t)$, $r_3(t)$ and $r_4(t)$ with t in $[-1, 1]$, and their local implicit approximations and local explicit approximation are shown in Figures 3.2, 3.3, 3.4 and 3.5. Note the good quality of local implicit approximation. Tables 3.1, 3.2 and 3.3, for $r_1(t)$,$r_2(t)$ and $r_3(t)$ respectively, list the y-values of a sequence of x-values to quantify how accurately the low degree local implicit forms approximate the original curves. The corresponding values of local explicit forms are also listed for comparison. We observe that

(a) For local implicit approximation, $d(t, n + k) < d(t, n)$ for t in $[-1, 1]$ and $k \geq 1$. In addition, $T(\epsilon, n) < T(\epsilon, n + k)$ for $k \geq 1$.
(b) $T(\epsilon, 2)$ of local implicit approximation is greater than $T(\epsilon, 6)$ of local explicit approximation.
(c) Degree 2 and 3 local implicit approximations give very accurate approximations on a reasonable range of t.
(d) Degree 5 local implicit approximation approximates the original curve very precisely at least for $-1 \leq t \leq 1$. □

Example 3.3 Local implicit approximations can be derived at singularities, including cusps, where local explicit approximation fails. Let $r_5(t) = (5t^3 + 2t^2, t^4 - 3t^3 + 2t^2)$ with the implicit form $f^4(x, y) = -x^4 + 55x^3 + 683x^2y + 1325xy^2 + 625y^3 - 336x^2 + 672xy - 336y^2$. The origin is a cusp of $r(t)$ with tangent $x - y = 0$. The degree 2 local implicit approximation is a double line $(x - y)^2$, which is the best degree 2 approximation one can derive at cusp. The degree 3 local implicit approximation is $x^2 - 2xy + y^2 - 0.16259766x^3 - 2.0356445x^2y - 3.940918xy^2 - 1.8608398y^3$ which shows very nice approximation to the $r(t)$ with t in $[-1, 1]$, see Figure 3.6. Figure 3.7 shows the degree 3 and degree 4 local implicit approximations of $r_6(t) = ((5t^5 - 16t^4 + 10t^3 + 4t^2)/w(t), (t^5 + t^4 + 2t^3 - 16t^2)/w(t))$, where $w(t) = 0.1t^3 + 0.1t^2 - 2t + 12.5$. The $r_6(t)$ is a singular curve with origin as a cusp and a self-intersection point. The degree 4 local implicit approximation shows remarkable performance. Note that the degree i, $i > 1$, local implicit approximation of $r_7(t) = (t^2, t^5)$, with implicit form $x^5 - y^2 = 0$, is the tangent line of $r_7(t)$ at the origin. $\square$

3.4 Discussion

A good local approximation of a curve would be the one that will show more accurate local approximation and larger $T(\epsilon, n)$ for a given ϵ when n, the degree of approximation, increases. The local implicit approximation $g^n(x, y) = 0$, $n < m$, of $r(t)$ is determined by $\varphi(n)$ linear conditions imposed on its coefficients, where $\varphi(n)$ is the degree of freedom of $g^n(x, y)$. Thus, when n increases, the raised degree of freedom requires more conditions to be satisfied in order to determine the coefficients of $g^n(x, y)$ and finally determines the exact implicitization when $n = m$. Hence local implicit approximation is capable of approximating a given curve not only locally but also globally in the sense that $T(\epsilon, n)$, for a given ϵ, will be larger when n increases. On the other hand, a local explicit approximation is limited because of the Inverse Function Theorem. That is, an explicit form approximates the given curve only locally for $|x| < R$, where R is its radius of convergence, and it diverges for $|x| \geq R$.

When computing an explicit approximation $y = h(x)$ directly from the curve $r(t)$, we first compute the degree n power series $t = \sum_{i=1}^{n} d_i x^i$ from $x = x(t)$, and then substitute it for t in $y = y(t)$. As a result, only the first n coefficients of $h(x)$ are exact and the remaining coefficients obtained in the computation should be discarded. Moreover, substituting $t = \sum_{i=1}^{n} d_i x^i$ for t in $y = y(t)$ is not a cheap computation, especially for high degree local explicit approximations. Hence, the computation of a local explicit approximations of a parametric curve directly from $r(t)$ is more costly than from the implicit form. In general, the computation of local implicit approximation involves generating the α_i^n and solving the linear system, which is fairly efficient for low-degree approximation.

The local explicit approximation $y = h(x) = \sum_{i=1}^{n} e_i x^i$ of $r(t)$ is an analytic function which is never singular at the origin and hence it can only describe a regular branch at the origin. Since for $r(t)$ we have separated the branches through the parameterization, and in the implicit form $f^m(x, y)$ we have not, the local explicit approximation of $f^m(x, y)$ does not exist at any singularity and the computation of the local explicit approximation of $r(t)$ fails only at singularities of an individual branch, i.e. at cusps and flat points. In contrast, the local implicit approximation always exists.

4 Local Implicit Approximation of Parametric Surfaces

A rational parametric surfaces can be defined by rational polynomials in two parameters. Let

$$P(s,t) = (x(s,t), y(s,t), z(s,t)) = \left(\frac{p(s,t)}{w(s,t)}, \frac{q(s,t)}{w(s,t)}, \frac{r(s,t)}{w(s,t)}\right)$$

be a rational parametric surface of degree m containing the origin, where

$$p(s,t) = \sum_{i+j=1}^{m} a_{ij} s^i t^j, \qquad q(s,t) = \sum_{i+j=1}^{m} b_{ij} s^i t^j,$$

$$r(s,t) = \sum_{i+j=1}^{m} c_{ij} s^i t^j, \qquad w(s,t) = \sum_{i+j=0}^{m} d_{ij} s^i t^j$$

with a_{ij}, b_{ij}, c_{ij}, $i+j=m$, not all zeros and $d_{00} \neq 0$. It has been shown by Sederberg [16] that a parametric surface defined as the above forms has an irreducible implicit forms $f^d(x,y,z) = 0$ of degree $d \leq m^2$ in x, y and z. The implicit forms can be found by eliminating s and t from

$$x - x(s,t) = 0, \qquad y - y(s,t) = 0, \qquad z - z(s,t) = 0$$

However, the elimination is computationally expensive and may yield extraneous factors. It is thus desirable to have a local implicit approximation that approximates the parametric surface patch locally and has degree much smaller than d.

Montaudouin et al. [11] proposed a local explicit approximation for *integral* parametric surfaces $P(s,t)$, i.e. $w(s,t) = 1$, by using the Inverse Function theorem, and obtaining $z = z(s(x,y), t(x,y)) = \sum_{i+j=1}^{nm} \gamma_{ij} x^i y^j$. Formulae for computing $s(x,y)$ and $t(x,y)$ were given that involve solving a series of two by two linear systems. Note that, on computing the coefficients of degree n local explicit approximation directly from $P(s,t)$ as above, the resulting γ_{ij} are exact only for $1 \leq i+j \leq n$, the remaining γ_{ij} should be discarded. The quality of this approximation can be judged by substituting $x(s,t), y(s,t)$ and $z(s,t)$ for x, y and z respectively in

$$f(x(s,t), y(s,t), z(s,t)) = z(s,t) - \sum_{i+j=1}^{n} \gamma_{ij} (x(s,t))^i (y(s,t))^j = \sum_{i+j=n+1}^{nm} \delta_{ij} s^i t^j$$

which is generally a n-th order of contact.

We propose a local implicit approximation of $P(s,t)$ of the form

$$g(x,y,z)=0$$

with g having degree $n<m$. This method works both for integral and rational parametric surfaces.

4.1 Derivation of the Local Implicit Approximation

Let $g^n(x,y,z)=\sum_{i+j+k=1}^{n} e_{ijk}x^iy^jz^k$, $n<m$, be a general implicit form of degree n surface that passes $(0,0,0)$. $g^n(x,y,z)$ has $\rho'(n)=((n+1)(n+2)(n+3)-6)/6$ terms with one coefficient for each term. Thus $g^n(x,y,z)$ has $\rho(n)=\rho'(n)-1$ degrees of freedom since $g^n(x,y,z)=0$ is unique up to a constant fator. Hence the coefficients of $g^n(x,y,z)$ can be determined by $\rho(n)$ independent linear conditions imposed on its coefficients.

When substituting $x(s,t),y(s,t)$ and $z(s,t)$ into $g^n(x,y,z)$, we have

$$g^n(x(s,t),y(s,t),z(s,t))=\frac{G^n(p(s,t),q(s,t),r(s,t),w(s,t))}{(w(s,t))^n}=\frac{\sum_{i+j=1}^{nm}\alpha_{ij}^n s^it^j}{\sum_{i+j=0}^{nm}\beta_{ij}s^it^j}$$

where $G^n(x,y,z,w)$ is the homogeneous form of $g^n(x,y,z)$, and α_{ij}^n is a linear polynomial of e_{ijk}, $1\le i+j+k\le n$. The local implicit approximation $g^n(x,y,z)$ of parametric surface $P(s,t)$ is computed as in the case of curve approximation. The following section shows the recursive way for deriving α_{ij}^n.

4.1.1 Computing α_{ij}^n

Let α_{ij}^{n-1} be the coefficient of s^it^j in $G^{n-1}(p(s,t),q(s,t),r(s,t),w(s,t))$. Then

$\alpha_{ij}^n=$ *coefficient of* s^it^j *in*

$$(w(s,t)\sum_{i+j=1}^{m(n-1)}\alpha_{ij}^{n-1}s^it^j+\sum_{k_1+k_2+k_3=n}e_{k_1k_2k_3}(p(s,t))^{k_1}(q(s,t))^{k_2}(r(s,t))^{k_3})$$

By setting $(p(s,t))^k = (\sum_{i+j=1}^{m} a_{ij}s^it^j)^k = \sum_{i+j=k}^{km}(a(k))_{ij}s^it^j$ and similarly for $(q(s,t))^k = \sum_{i+j=k}^{km}(b(k))_{ij}s^it^j$ and $(r(s,t))^k = \sum_{i+j=k}^{km}(c(k))_{ij}s^it^j$ we have

$$\begin{aligned}\alpha_{ij}^n &= \sum_{l_1=1}^{i}\sum_{l_2=1}^{j} \alpha_{l_1l_2}^{n-1} d_{(i-l_1)(j-l_2)} \\ &\quad + \sum_{k_1+k_2+k_3=n} \sum_{\substack{p_1+p_2+p_3=i \\ q_1+q_2+q_3=j}} e_{k_1k_2k_3}(a(k_1))_{p_1q_1}(b(k_2))_{p_2q_2}(c(k_3))_{p_3q_3}\end{aligned}$$

and in the base case, $\alpha_{ij}^1 = e_{100}a_{ij} + e_{010}b_{ij} + e_{001}c_{ij}$.

For an integral parametric surface $P(s,t)$, since $(a(k))_{ij} = 0, (b(k))_{ij} = 0$ and $(c(k))_{ij} = 0$ for $i+j<k$ and $\alpha_{ij}^n = 0$ for $i+j>(n-1)m$, we have for $1 \le i+j \le n-1$,

$$\alpha_{ij}^n = \alpha_{ij}^{n-1}$$

for $n \le i+j \le (n-1)m$,

$$\alpha_{ij}^n = \alpha_{ij}^{n-1} + \sum_{k_1+k_2+k_3=n} \sum_{\substack{p_1+p_2+p_3=i \\ q_1+q_2+q_3=j}} e_{k_1k_2k_3}(a(k_1))_{p_1q_1}(b(k_2))_{p_2q_2}(c(k_3))_{p_3q_3}$$

for $(n-1)m < i+j \le nm$,

$$\alpha_{ij}^n = \sum_{k_1+k_2+k_3=n} \sum_{\substack{p_1+p_2+p_3=i \\ q_1+q_2+q_3=j}} e_{k_1k_2k_3}(a(k_1))_{p_1q_1}(b(k_2))_{p_2q_2}(c(k_3))_{p_3q_3}$$

$(a(k))_{ij}, (b(k))_{ij}$ and $(c(k))_{ij}$ can be computed recursively as shown in [11].

4.1.2 Computing the Local Implicit Approximation of Surfaces

We define matrix $\mathbf{A}_{mn}$ to be the coefficient matrix of $(\alpha_{10}^n, \alpha_{01}^n, \alpha_{20}^n, \alpha_{11}^n, \alpha_{02}^n, \ldots, \alpha_{(nm)0}^n, \alpha_{(nm-1)1}^n, \ldots, \alpha_{1(nm-1)}^n, \alpha_{0(nm)}^n)^T$ with respect to the vector of variables $\mathbf{e}_n = (e_{100}, e_{010}, e_{001}, e_{200}, e_{110}, e_{101}, e_{020}, e_{011}, e_{002}, \ldots, e_{00n})^T$. Hence $\mathbf{A}_{mn}$ is of dimension $((nm+1)(nm+2))/2 - 1$ by $\rho(n)+1$, and

$$\mathbf{A}_{mn}\mathbf{e}_n = (\alpha_{10}^n, \alpha_{01}^n, \alpha_{20}^n, \alpha_{11}^n, \alpha_{02}^n, \ldots, \alpha_{(nm)0}^n, \alpha_{(nm-1)1}^n, \ldots, \alpha_{1(nm-1)}^n, \alpha_{0(nm)}^n)^T$$

As in the curve case, the rank of $\mathbf{A}_{mn}$ is critical for solving for the unknown coefficients of the local implicit approximation $g^n(x,y,z)$. The following theorem characterizes the rank of $\mathbf{A}_{mn}$.

Theorem 4.1 *If $P(s,t)$ is a properly parameterized degree m rational parametric surface with irreducible implicit form $f^d(x,y,z)=0$, $d=m^2$, then*

$$rank(\mathrm{A}_{mn}) = \begin{cases} \rho(n)+1 & if\ n<d \\ \rho(n) & if\ n=d \end{cases}$$

Proof: Similar to proofs of Lemma 3.2 and Lemma 3.3. □

As a result of Theorem 4.1, it is clear that the exact implicitization of $P(s,t)$ is the solution of $\mathrm{A}_{mm}\mathrm{e}_m = 0$, with one variable fixed. When computing $g^n(x,y,z)$ at nonsingular point $(0,0)$, let $\mathrm{B}_{mn}\mathrm{e}_n = \mathrm{b}_{mn}$ denote the system of linear equations $\alpha=0$, $\alpha_{10}^n=0$, $\alpha_{01}^n=0$, $\alpha_{20}^n=0$, $\alpha_{11}^n=0$, $\alpha_{02}^n=0, \ldots, \alpha_{(nm)0}^n=0$, $\alpha_{(nm-1)1}^n=0, \ldots, \alpha_{1(nm-1)}^n=0$, $\alpha_{0(nm)}^n=0$, where $\alpha=0$ is $e_{100}-1=0$, $e_{010}=0$, or $e_{001}-1=0$ depending on the gradient of the surface at the origin. Let $\mathrm{B}_{mn}^i\mathrm{e}_n=\mathrm{b}_{mn}^i$ be the subsystem that consists of the first $i+1$ equations of $\mathrm{B}_{mn}\mathrm{e}_n=\mathrm{b}_{mn}$. Now the coefficients of $g^n(x,y,z)$ are the solution of

$$\mathrm{B}_{mn}^k\mathrm{e}_n=\mathrm{b}_{mn}^k$$

where $k\geq\rho(n)$ is the smallest integer such that B_{mn}^k is of rank $\rho(n)+1$ and $\mathrm{B}_{mn}^k\mathrm{e}_n=\mathrm{b}_{mn}^k$ is consistent. $\mathrm{B}_{mn}^k\mathrm{e}_n=\mathrm{b}_{mn}^k$ may be inconsistent. If this happens, some equations must be removed from $\mathrm{B}_{mn}^k\mathrm{e}_n=\mathrm{b}_{mn}^k$ to ensure the consistency.

One alternative of handling the inconsistencies is that we replace $\alpha=0$ in $\mathrm{B}_{mn}\mathrm{e}_n=\mathrm{b}_{mn}$ with $e_{n00}-1=0$, $e_{0n0}=0$, or $e_{00n}-1=0$ and then solve it as usual. Experiments show that $g^n(x,y,z)$ computed by this method can be of the form $(ax+by+cz)^n$ for some a,b,c, i.e. it degenerates to the tangent plane. To remove the degeneracy, we do the following:

1. Solve for $g^1(x,y,z)$, and compute

$$\sum_{i+j=n}\beta_{ij}s^it^j=(g^1(x(s,t),y(s,t),z(s,t)))^n$$

2. Consider $\mathrm{B}_{mn}^{\bar{k}}\mathrm{e}_n=\mathrm{b}_{mn}^{\bar{k}}$, where $\bar{k}\geq\rho(n)$ is the smallest integer such that $\mathrm{B}_{mn}^{\bar{k}}$ is of rank $\rho(n)$.
3. Find a β_{ij} which is nonzero and augment the corresponding $\alpha_{ij}=0$ to the system $\mathrm{B}_{mn}^{\bar{k}}\mathrm{e}_n=\mathrm{b}_{mn}^{\bar{k}}$, then solve it.

This computation of the local implicit approximation results in an approximately $n^{3/2}$-th order of contact, which is not as high as that of curve case.

Example 4.1
Consider $P(s,t) = (x(s,t)/w(s,t), y(s,t)/w(s,t), z(s,t)/w(s,t))$ where

$$\begin{aligned}
x(s,t) &= -200t^2 + 12st + 400t - 200s^2 - 10s \\
y(s,t) &= 15t^2 - 14st + 10t - 11s^2 + 400s \\
z(s,t) &= 200t^2 + 11st - t + 200s^2 + 2s \\
w(s,t) &= 100t^2 - 200t + 100s^2 + 200
\end{aligned}$$

We compute degree 2 and degree 3 local implicit approximations

$$\begin{aligned}
&g^2(x,y,z) = \\
&\quad -108.44294z^2 - 10.264638yz - 13.162097xz + 381.19047z \\
&\quad - 95.092836y^2 - 5.241114xy - 1.8809524y - 94.85476x^2 + x
\end{aligned}$$

and

$$\begin{aligned}
&g^3(x,y,z) = \\
&\quad 1.3012126z^3 + 5.16125yz^2 - 46.69081xz^2 - 103.818164z^2 \\
&\quad - 1.1158845y^2z + 15.622598xyz + 4.518084yz \\
&\quad + 1.267575x^2z + 180.40466xz + 381.19047z - 3.6884814y^3 \\
&\quad - 48.00386xy^2 - 95.16589y^2 + 5.5613696x^2y - 6.1573525xy \\
&\quad - 1.8809524y - 44.977395x^3 - 94.34699x^2 + x
\end{aligned}$$

Note that the normal of $f^4(x,y,z)$, the exact implicit form of $P(s,t)$, at the origin is almost parallel to z-axis. Thus, to show the performance of the local implicit approximation, we intersect cylinder $h(x,y,z) = x^2 + y^2 - r^2 = 0$ with surfaces f^4, g^2, and g^3, and plot the intersection curves of $f^4 = 0 \cap h = 0, g^3 = 0 \cap h = 0$, and $g^2 = 0 \cap h = 0$ in one figure. Figures 4.1(a) and 4.1(b), for $r = 0.25, 0.5, 0.75, 1.00, 1.25, 1.50$, and 1.75, show the intersection curves in cylindrical coordinates. Table 4.1 lists the maximal deviations between the intersection curves $f^4 = 0 \cap h = 0$ and $g^3 = 0 \cap h = 0$, and $f^4 = 0 \cap h = 0$ and $g^2 = 0 \cap h = 0$. □

4.2 Computing Surfaces Intersection

We apply the local implicit approximation as follows to surface intersection. Using subdivision, we locate an initial point on each segment of the intersection curve. Once the initial point is found, local implicit approximation can be used to obtain local approximations to the intersection curve at curve points either for stepping in the curve tracing or for constructing a piecewise algebraic curve approximation of the intersection curve. Let $P_1(s,t) = (x_1(s,t), y_1(s,t), z_1(s,t))$ and $P_2(u,v) = (x_2(u,v), y_2(u,v), z_2(u,v))$ be two given parametric surfaces that both contain point p. Substituting $x_2(u,v)$, $y_2(u,v)$ and $z_2(u,v)$ into the local implicit approximation of P_1 at point p yields $Q(u,v) = 0$, which locally approximates the intersection curve at point p. Instead of tracing intersection space curve, we trace along $Q(u,v) = 0$ from point p until the deviation between point computed and the intersection curve exceeds a pre-specified tolerance. The same procedure proceeds until the end of intersection curve is reached. Thus a sequence of low-degree algebraic plane curve segments which approximate the intersection space curve can be computed while tracing the intersection curve. Furthermore, each algebraic plane curve segment can be approximated by a piecewise C^0 rational parametric approximation, see [18].

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	x value								
	-5.0	-3.0	-2.0	-1.25	0	1.25	2.0	4.0	7.0
exact	-11.845579	-6.5811195	-4.2442303	-2.5918415	0	2.4172442	3.793271	7.213065	11.663555
li 4	-11.845584	-6.5811195	-4.2442300	-2.5918415	0	2.4172442	3.793271	7.213066	11.663522
li 3	-11.843186	-6.5811080	-4.2442300	-2.5918415	0	2.4172442	3.793271	7.213022	11.658528
li 2	-11.910466	-6.5846540	-4.2446346	-2.5918765	0	2.4172716	3.793540	7.220462	11.761091
le 2	-23.889015	-11.000046	-6.2222424	-3.3680634	0	1.6319366	1.777758	-0.8859694	-13.222469
le 4	-24.304024	-11.078828	-6.2439766	-3.3730750	0	1.6359663	1.793057	-0.79231834	-12.91141
le 6	-24.336723	-11.081027	-6.2442436	-3.3730989	0	1.6359847	1.793233	-0.7881178	-12.878107
le 12	-25.297968	-11.082955	-6.2442613	-3.3730989	0	1.6359840	1.793167	-0.9260575	-103.35114

Table 3.1

	x value							
	-0.25	-0.1	0.0	0.1	0.25	1.0	1.25	1.5
exact	-0.4291671 -1.1245294	-0.11828928 -1.4897133	0 -1.6495429	0.08622920 -1.7816917	0.17517897 -1.9479930	0.17769729 -2.5130520	0.043922530 -2.6506336	-0.16892694 -2.7737687
li 4	-0.4291671 -1.1245286	-0.11828928 -1.4897133	0 -1.6495163	0.08622920 -1.7816285	0.17517897 -1.9478126	0.17769726 -2.5069103	0.043921884 -2.6342275	-0.168934 -2.7259927
li 3	-0.4291675 -1.1223613	-0.11828928 -1.4540809	0 -1.6388266	0.08622920 -1.7638947	0.17517897 -1.9152321	0.17769955 -2.2900155	0.043549964 -2.2978785	-0.16974114 -2.2328222
li 2	-0.4286827 -1.1285945	-0.11826820 -1.4734732	0 -1.6147509	0.08622874 -1.7239690	0.17514941 -1.8473737	0.16556032 -2.0102053	0.008065259 -1.91015380	-0.27427490 -1.685317
le 12	-0.42583282 ——	-0.11828927 ——	0 ——	0.0862292 ——	0.17486227 ——	-9337.529 ——	-143129.16 ——	-1323236.8 ——
le 6	-0.41469532 ——	-0.11827987 ——	0 ——	0.0862238 ——	0.17268895 ——	-18.46853 ——	-75.29226 ——	-234.37852 ——
le 4	-0.39622853 ——	-0.11815020 ——	0 ——	0.0861461 ——	0.16890110 ——	-2.872428 ——	-7.885883 ——	-17.354166 ——
le 2	-0.34722220 ——	-0.11555555 ——	0 ——	0.0844444 ——	0.15277778 ——	0.55555556 ——	-1.1805556 ——	-2.0 ——

Table 3.2

	x value								
	-3.5	-2.5	-0.5	-0.25	0	0.25	0.5	2.0	2.5
exact	3.1174486	2.1534863	.45135155	.22867906	0	-.23612773	-.48143223	-2.3418179	-3.4202664
li 4	3.1174495	2.1534863	.45135155	.22867906	0	-.23612773	-.48143226	-2.3418179	-3.4202664
li 3	3.1159813	2.1534630	.45135158	.22867908	0	-.23612775	-.48143230	-2.3418021	-3.4196750
li 2	2.9099880	2.1148510	.45132570	.22867821	0	-.23612662	-.48139048	-2.2394760	-2.0944711
le 12	2.7423260	2.1480120	.45135152	.22867905	0	-.23612773	-.48143230	-2.3401886	-3.3552332
le 6	2.8504105	2.1270490	.45135110	.22867908	0	-.23612775	-.48143163	-2.3187237	-3.2120173
le 4	2.7878397	2.0934718	.45132630	.22867824	0	-.23612681	-.48139983	-2.2753644	-3.0587234
le 2	2.5223215	1.9501640	.44943514	.22843071	0	-.23585550	-.47913630	-2.0947520	-2.6926932

Table 3.3

degree	r=0.25	r=0.50	r=0.75	r=1.25	r=1.50	r=1.75
li 3	0.000000	0.000004	0.000035	0.000837	0.004964	0.019612
li 2	0.000289	0.002531	0.009630	0.066627	——	——

Table 4.1

li n : local implicit form of degree n. le n : local explicit form of degree n.

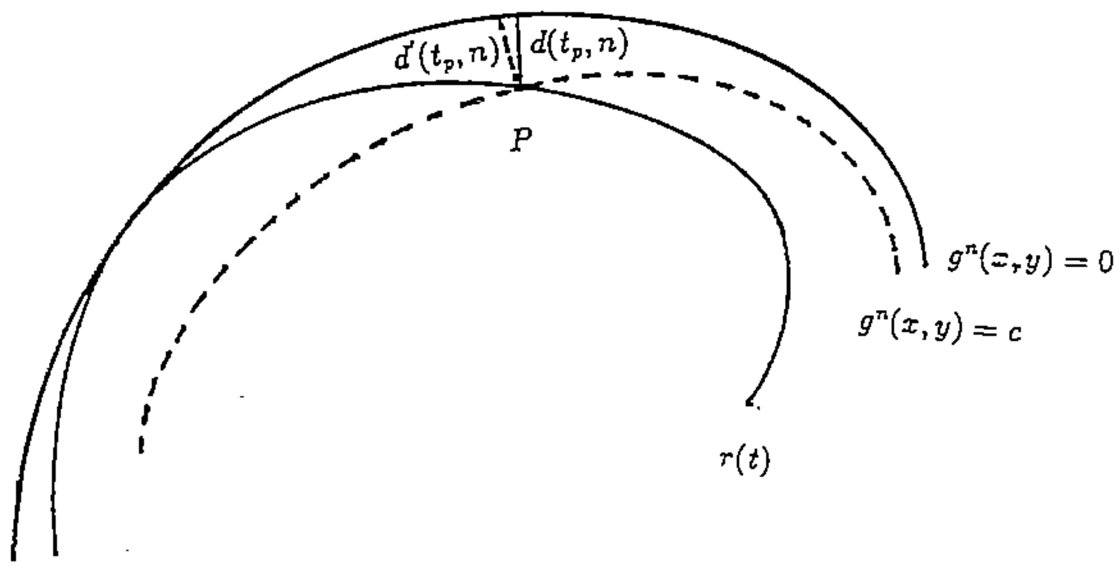


Figure 3.1

Figure 3.2

$$r_1(t) = (t^6 + t^5 - 2t^3 + 3t^2 + 12t, t^6 - t^5 + t^4 - 4t^3 - 2t^2 + 24t)$$

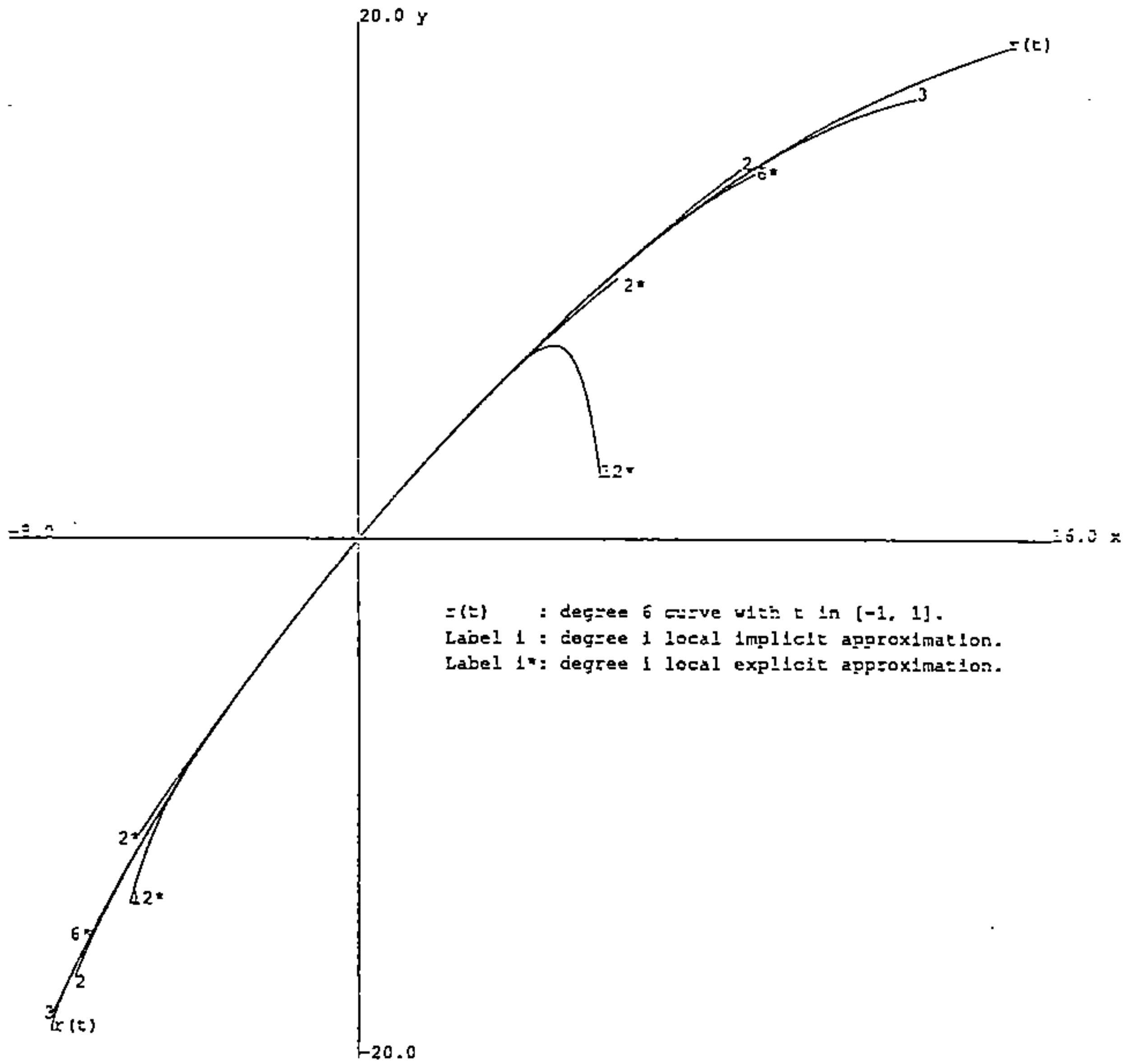


Figure 3.3

$$r_2(t) = (3t^6 - 4t^5 - 8t^3 + 6t^2 + 3t, -3t^6 + 4t^5 + 5t^4 - 6t^3 - 8t^2 + 3t)$$

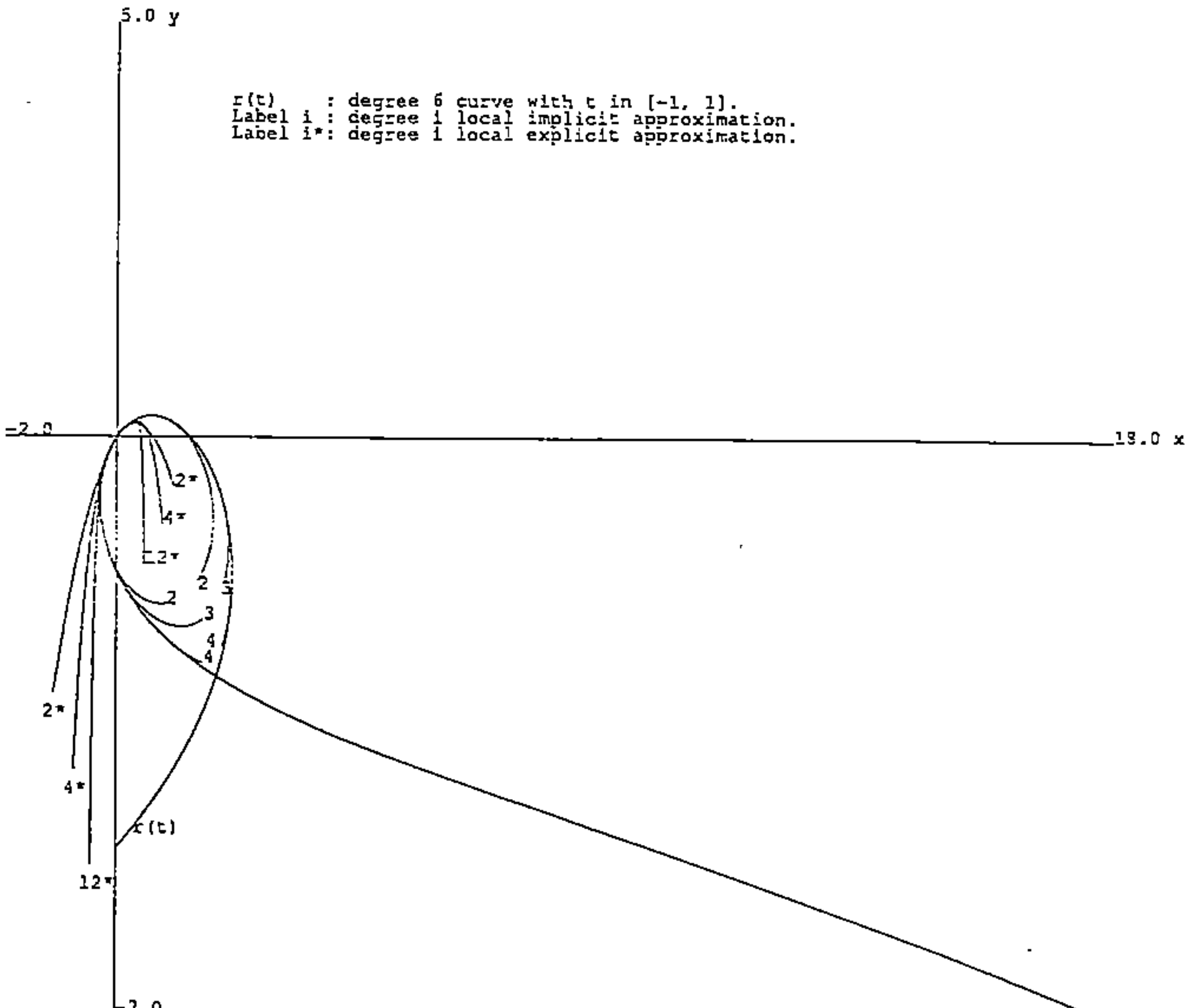


Figure 3.4

$$r_3(t) = (3t^6 + t^5 - 2t^4 + 38t^3 - 5t^2 - 14t, t^6 - 12t^5 - 2t^4 + 2t^3 - 7t^2 + 13t)$$

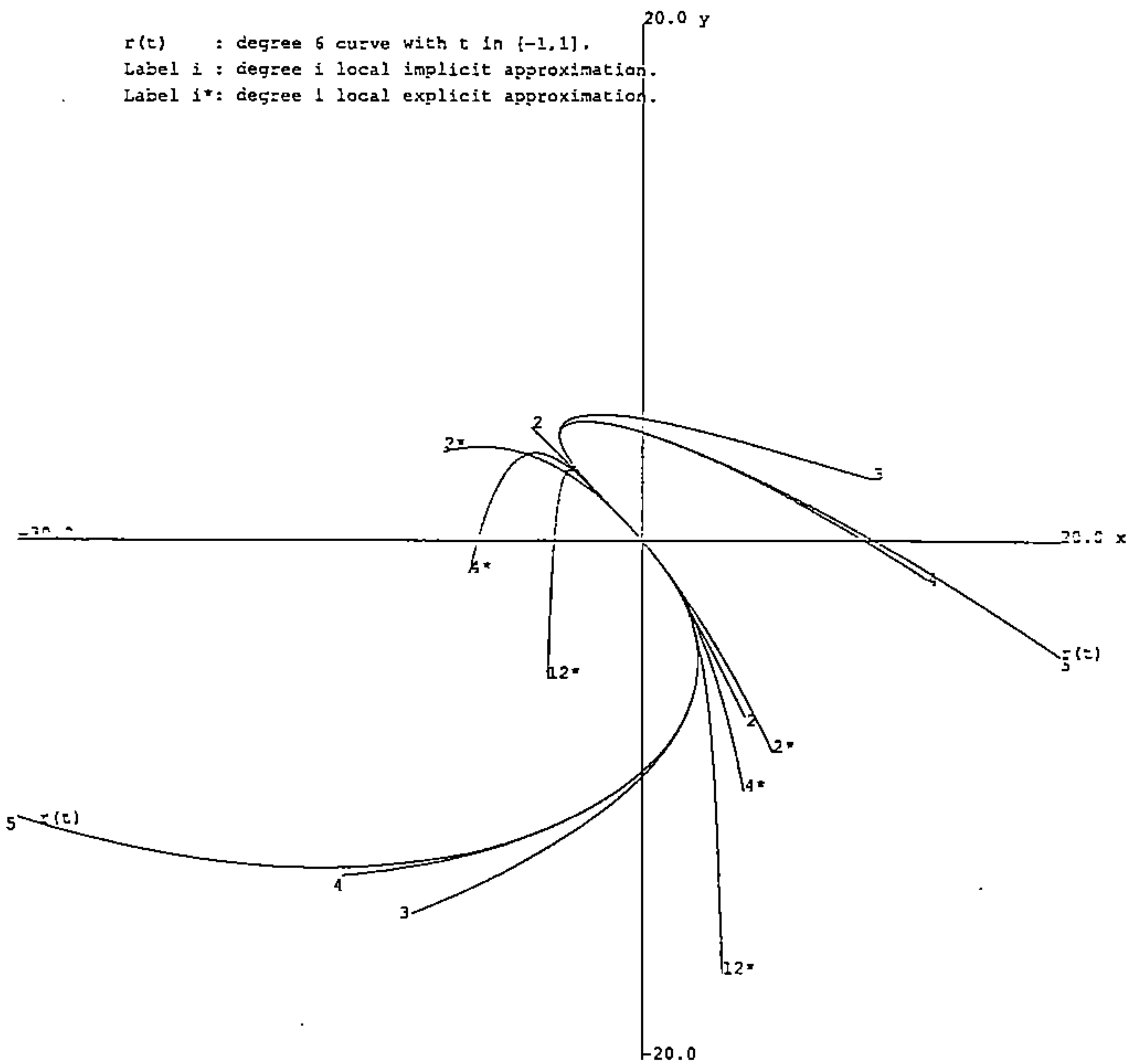


Figure 3.5

$$r_4(t) = \left(\frac{t^6 + 3t^5 - 6t^4 + 4t^3 - 36t^2 + 36t}{7t^6 + 10t^5 + 9t^4 + 6t^2 + 3t + 7}, \frac{3t^6 + t^5 - 2t^4 + 39t^3 - 69t^2 + 33t}{7t^6 + 10t^5 + 9t^4 + 6t^2 + 3t + 7}\right)$$

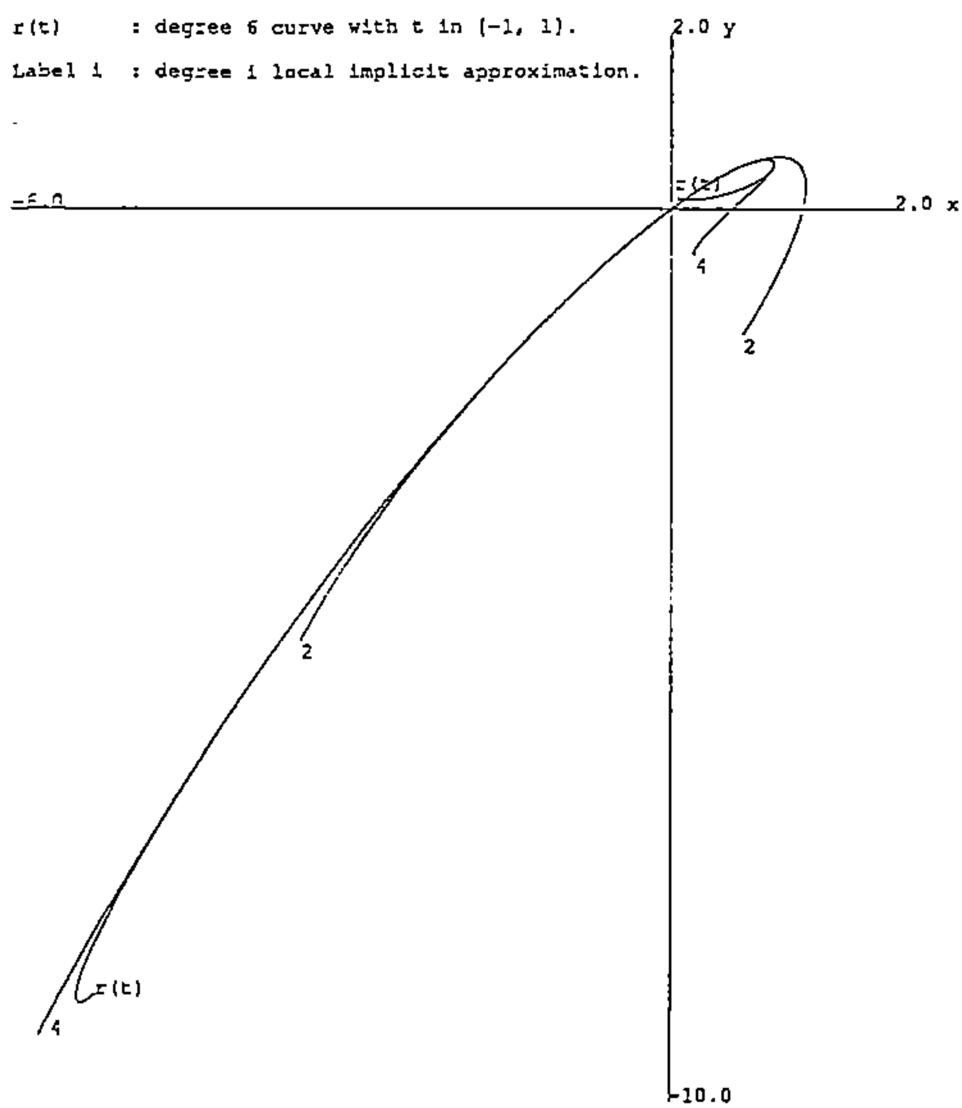


Figure 3.6

$$r_5(t) = (5t^3 + 2t^2, t^4 - 3t^3 + 2t^2)$$

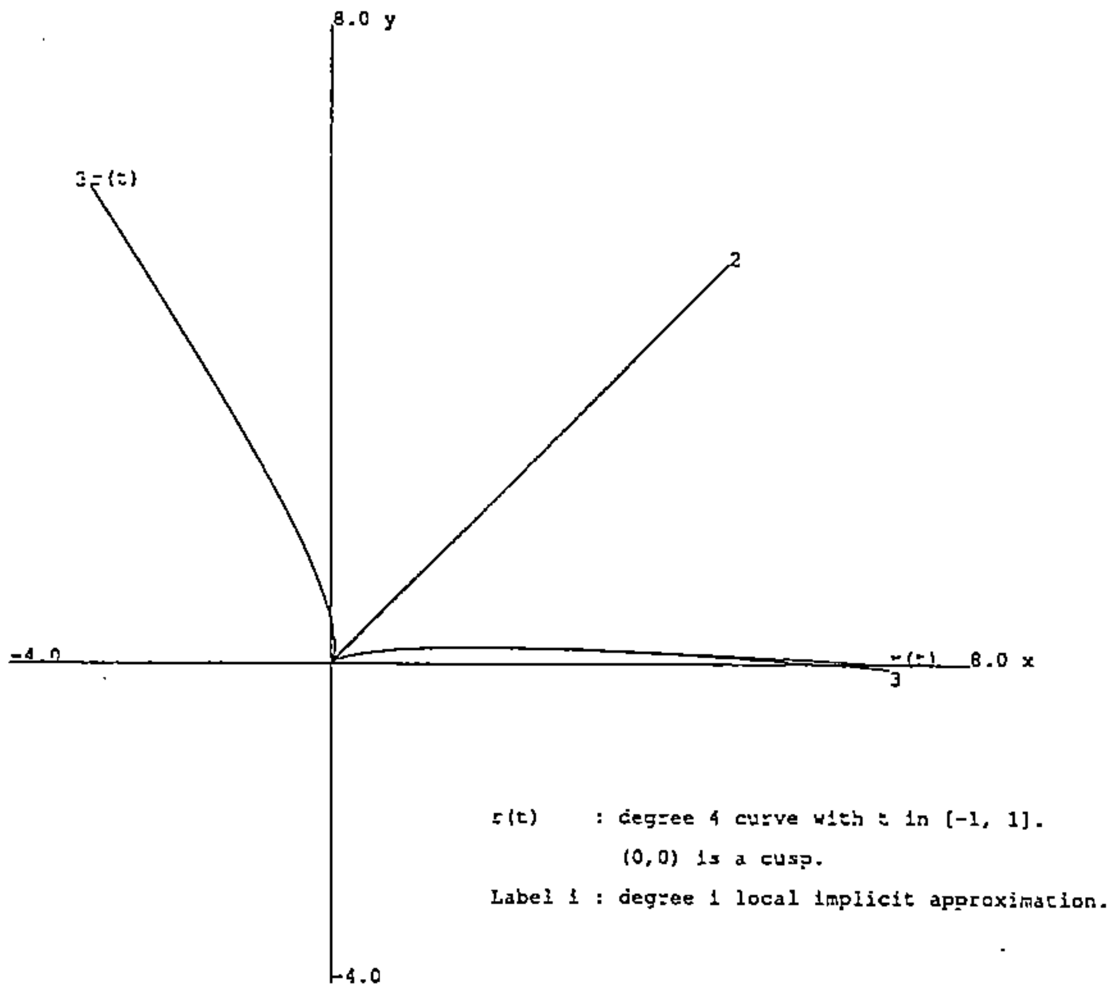
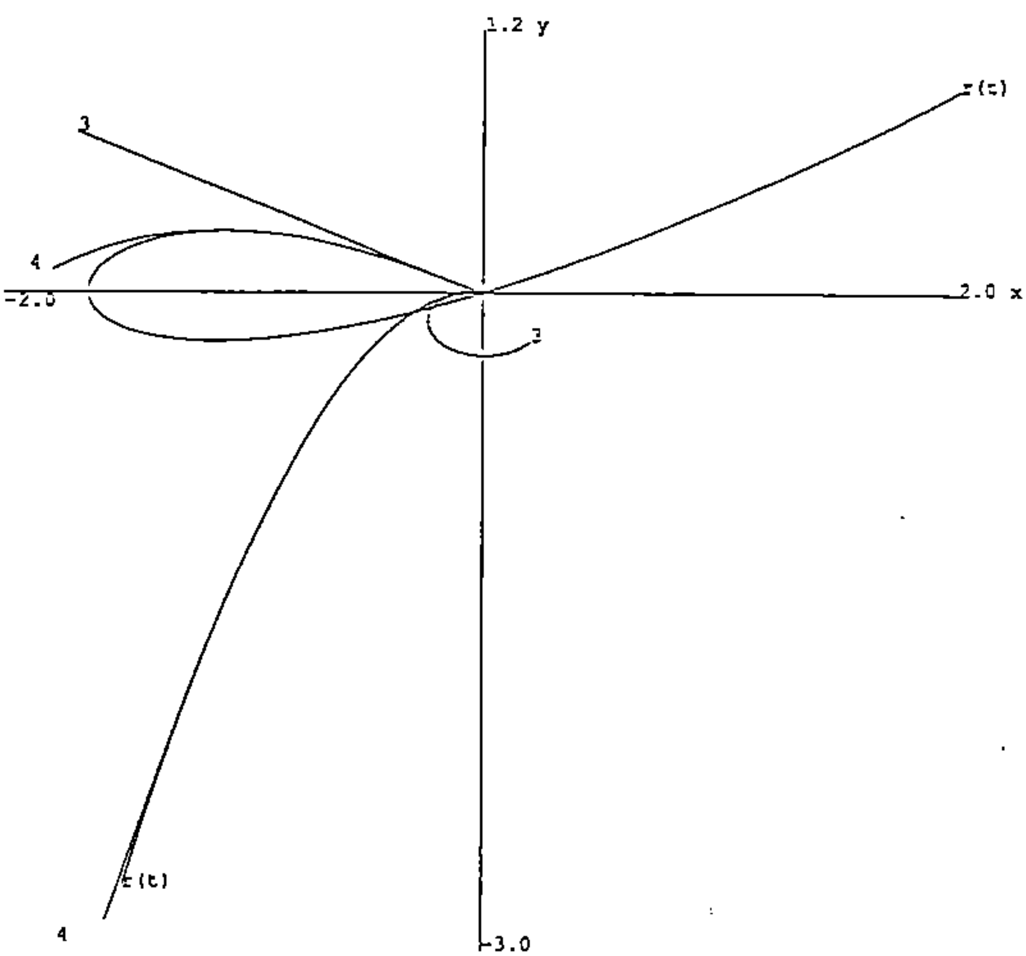


Figure 3.7

$$r_6(t) = (\frac{5t^5 - 16t^4 + 10t^3 + 4t^2}{0.1t^3 + 0.1t^2 - 2t + 12.5}, \frac{t^5 + t^4 + 2t^3 - 16t^2}{0.1t^3 + 0.1t^2 - 2t + 12.5})$$

```
r(t)      : degree 5 rational curve.
The origin is a cusp and a self-intersection point.
Label i   : degree i local implicit approximation.
```



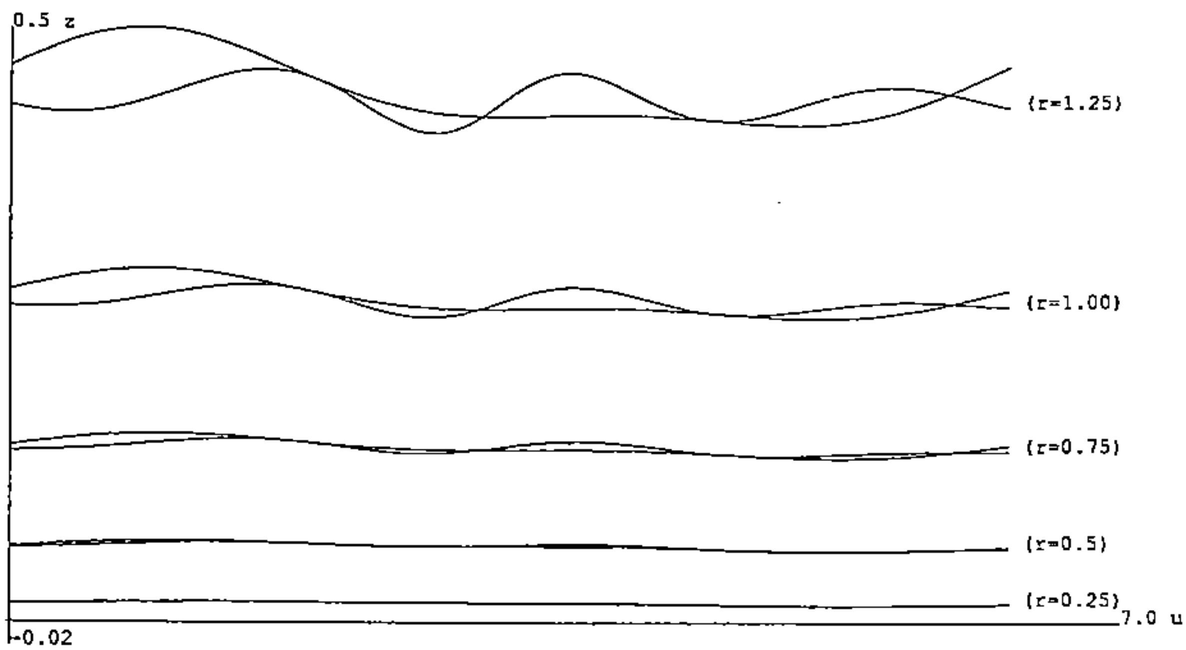


Figure 4.1(a) $f^4 = 0 \cap h = 0$ and $g^2 = 0 \cap h = 0$

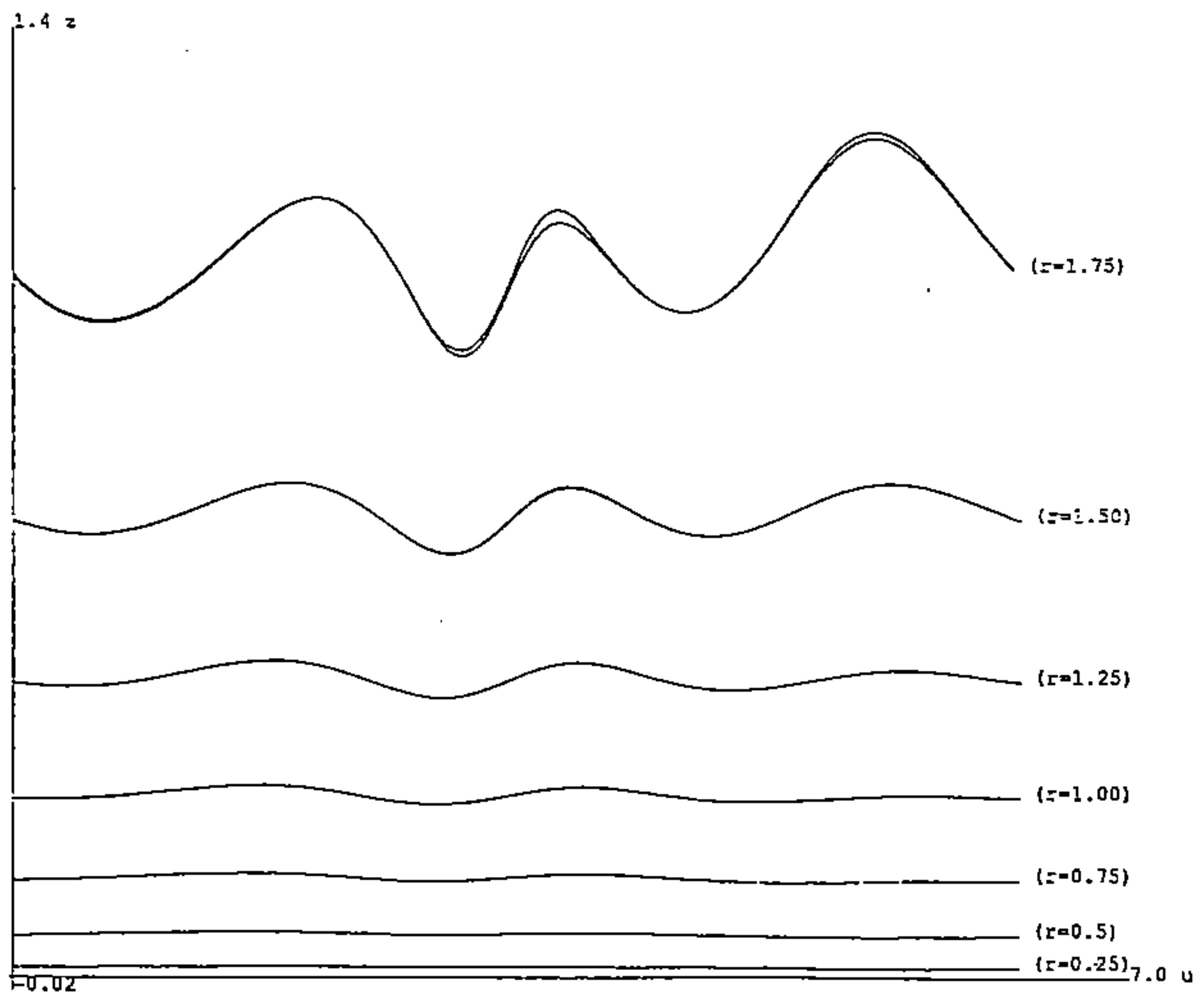


Figure 4.1(b) $f^4 = 0 \cap h = 0$ and $g^3 = 0 \cap h = 0$